Quantitative Literacy:
Thinking Between the Lines

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Chapter 8:
Voting and Social Choice

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Lesson Plan

- Measuring voting power: Does my vote count?
- Voting systems: How do we choose a winner?
- Fair division: What is a fair share?
- Apportionment: Am I represented?

Learning Objectives:

- Understand the relationship between a voter's power and the number of votes he or she has
- Measure true voting power:
  - Using the Banzhaf index
  - Using the Shapley-Shubik index

Chapter 8 Voting and Social Change
8.1 Measuring voting power: Does my vote count?

- **Voting Coalition:** A group of voters who vote the same way
- **Winning Coalition:** A set of voters with enough votes to determine the outcome of an election, otherwise it is a **Losing Coalition**
- **Quota:** The number of votes necessary to win the election, in a voting system
- **Example:** Suppose there are three delegates to a county convention: Abe has 4 votes from his precinct, Ben has 3 votes, and Condi has 1 vote. A simple majority of the votes wins.
  1. What is the quota?
  2. Make a table listing all of the coalitions of voters. Designate which of them are winning coalitions.

- **Solution:**
  1. There are 8 votes, so the quota for simple majority is 5 votes.
  2. The following table shows all possible coalitions and votes each coalition controls. The last column indicates whether the coalition controls a majority of the votes and is a winning coalition.

Number of Votes			Total Votes	Winning Coalition?
4	3	1		
Abe	Ben	Condi	8	Yes
Abe	Ben		7	Yes
Abe		Condi	5	Yes
Abe			4	No
	Ben	Condi	4	No
	Ben		3	No
		Condi	1	No

8.1 Measuring voting power: Does my vote count?

- **Critical voter:** A member of a winning coalition is a critical voter if the coalition becomes a losing coalition when that voter is removed.
- **Example:** The table below contains information from the three winning coalitions in the previous example.

Number of Votes			Total Votes	Winning
4	3	1		
Abe	Ben	Condi	8	Yes
Abe	Ben		7	Yes
Abe		Condi	5	Yes

- In the first coalition: Abe is the critical voter.
- In the second coalition: both are critical voters.
- In the third coalition: both are critical voters.

8.1 Measuring voting power: Does my vote count?

- The previous information can be summarized in the following *coalition table:*

TABLE 8.1 Coalition Table

Number of votes			Total votes	Winning coalition?	Critical voters
4	3	1			
Abe	Ben	Condi	8	Yes	Abe
Abe	Ben		7	Yes	Abe, Ben
Abe		Condi	5	Yes	Abe, Condi
Abe			4	No	Not applicable
	Ben	Condi	4	No	Not applicable
	Ben		3	No	Not applicable
		Condi	1	No	Not applicable

Winning Coalition and Critical Voters

A set of voters with enough voters to determine the outcome of an election is a *winning coalition*. A voter in a winning coalition is *critical* for that coalition if the coalition is no longer a winning one when that voter is removed. We can summarize the essential information about coalitions in a *coalition table*.

Counting coalitions: Number of coalitions

For n voters, there are $2^n - 1$ possible coalitions (each of which includes at least one voter).

- **Example:** If there are 7 voters, there are 2^7 possibilities for voters to be in or not in a coalition. This includes the possibility of all the voters not being in any coalition, so there are $2^7 - 1$ possibilities.

8.1 Measuring voting power: Does my vote count?

- **Banzhaf power index:** The number of times a voter is critical in a winning coalition divided by the total number of instances in which any voter is critical. Expressed as a fraction or percentage.
- **Example:** Use coalition Table 8.1 to compute the Banzhaf index for each county convention delegate described in the previous examples.
- **Solution:** From the coalition table: overall there were 5 instances in which any voter was critical. So the Banzhaf power index of a voter is the number of times that voter is critical divided by 5.
  - Abe was critical 3 of the 5 times, so he has a Banzhaf power index of 3/5 or 60%.
  - Ben was critical 1 of the 5 times, so he has a Banzhaf power index of 1/5 or 20%.
  - Condi was critical 1 of the 5 times. So she has a Banzhaf power index of 1/5 or 20%.

- **Example:** At the 1988 Democratic National Convention, Michael Dukakis's delegates had 1401 votes, Jesse Jackson's had 1218 votes, Al Gore's had 325 votes, and Bruce Babbitt's had 197 votes. It is common that candidates form coalitions. A simple majority is required to win the nomination.
  1. What is the quota?
  2. Determine the winning coalitions.
  3. Determine the critical voters (candidates) in each winning coalition.
  4. Determine the Banzhaf index of each candidate.

8.1 Measuring voting power: Does my vote count?

- **Solution:**
  1. The total number of votes is 3141, half that is 1570.5, so a simple majority is at least 1571 votes, and that is the quota.

Presidential candidate Jesse Jackson addressing the 1988 Democratic Convention.

- **Solution (cont.):**
  - 2. The following table lists the $2^4 - 1 = 15$ positive coalitions

Votes					
1401	1218	325	197	Total votes	Winning coalition?
D	J	G	B	3141	Yes
D	J	G		2944	Yes
D	J		B	2816	Yes
D	J			2619	Yes
D		G	B	1923	Yes
D		G		1726	Yes
D			B	1598	Yes
				1401	No
	J	G	B	1740	Yes
	J	G		1543	No
	J		B	1415	No
	J			1218	No
		G	B	522	No
		G		325	No
			B	197	No

- **Solution (cont.):**

3. The accompanying table lists the winning coalitions only, along with the critical voters in each case.

Winning Coalitions Only					
Votes					
1401	1218	325	197	Total votes	Critical voters
D	J	G	B	3141	None
D	J	G		2944	D
D	J		B	2816	D
D	J			2619	D, J
D		G	B	1923	D
D		G		1726	D, G
D			B	1598	D, B
	J	G	B	1740	J, G, B

- **Solution (cont.):**

4. There are 12 instances in which a voter or candidate is critical:
   - Dukakis is critical 6 of the 12 times, so his Banzhaf index is
   $$\frac{6}{12} = \frac{1}{2} \text{ or } 50\%$$
   - Jackson is critical 2 of the 12 times, so his Banzhaf index is
   $$\frac{2}{12} = \frac{1}{6} \text{ or } 16.67\%$$
   - Gore is critical 2 of the 12 times, so his Banzhaf index is
   $$\frac{2}{12} = \frac{1}{6} \text{ or } 16.67\%$$
   - Babbitt is critical 2 of the 12 times, so his Banzhaf index is
   $$\frac{2}{12} = \frac{1}{6} \text{ or } 16.67\%$$

- **Benzhaf Power Index:** A voter's *Benzhaf index* is the number of times that voter is critical in some winning coalition divided by the total number of instances in which any voter is critical. The index is expressed as a fraction or as a percentage.

- **Swing voter:** Supposing the voters vote in order and their votes are added as they vote, **the swing voter** is the voter whose votes make the total meet the quota and thus decide the outcome. Which is the swing voter depends on the order the votes are cast.

- **Example:** Members of the European Union have votes on the Council determined roughly by a country's population but progressively weighted in favor of smaller countries. Ireland has 7 votes, Cyprus has 4 votes, and Malta has 3 votes. Supposing a simple majority wins, make a table with all the permutations of voters and the swing voter in each case.

- **Solution:** There are 14 votes total so a simple majority is 8. So the quota is 8. There are *n*! different permutations of *n* objects, where

$$n! = n \times (n-1) \times \cdots \times 1$$

So the 3 objects have $3! = 3 \times 2 \times 1 = 6$ permutations. The following table lists the permutations and swing voter in each case.

- **Solution (cont.):**

Order of Voters			Swing Voter
Ireland (7)	Cyprus (4)	Malta (3)	Cyprus
Ireland (7)	Malta (3)	Cyprus (4)	Malta
Cyprus (4)	Ireland (7)	Malta (3)	Ireland
Cyprus (4)	Malta (3)	Ireland (7)	Ireland
Malta (3)	Ireland (7)	Cyprus (4)	Ireland
Malta (3)	Cyprus (4)	Ireland (7)	Ireland

- **The Shapley-Shubik power index:** is calculated as the fraction (or percentage) of all permutations of the voters in which that voter is the swing.

- **Example:** Compute the Shapley-Shubik power index for the committee of Ireland, Cyprus, and Malta from the previous example.

- **Solution:** There are six permutations of the voters. Ireland is the swing in 4 of the 6 cases so the index for Ireland is $4/6 = 2/3$ or about 66.67%.
  - Cyprus is the swing in 1 of the 6 cases so its index is $1/6$ or 16.67%.
  - Malta also is the swing in 1 of the 6 cases so its index is $1/6$ or 16.67%.

- **Example:** In the 2004 election, the votes from Florida, Michigan, Ohio, and Pennsylvania were important in determining the outcome. Florida had 27 electoral votes, Michigan had 17, Ohio had 20, and Pennsylvania had 21. Assume a majority from only these four states would determine the election. In this case the quota is 43.
  1. How many permutations of these four states are there?
  2. Make a table listing each case and its swing voter.
  3. Find the Shapley-Shubik index for each state.
- **Solution:**
  1. The number of permutations of four items is
  $$4! = 4 \times 3 \times 2 \times 1 = 24$$
  2. The following table shows the different permutations and the swing vote state in each case.

- **Solution (cont.):**

Voters				Swing
FL (27)	MI (17)	OH (20)	PA (21)	MI
FL (27)	MI (17)	PA (21)	OH (20)	MI
FL (27)	MI (20)	MI (17)	PA (21)	OH
FL (27)	MI (20)	PA (21)	MI (17)	OH
FL (27)	MI (21)	MI (17)	OH (20)	PA
FL (27)	MI (21)	OH (20)	MI (17)	PA
MI (17)	FL (27)	OH (20)	PA (21)	FL
MI (17)	FL (27)	PA (21)	OH (20)	FL
MI (17)	OH (20)	FL (27)	PA (21)	FL
MI (17)	OH (20)	PA (21)	FL (27)	PA
MI (17)	PA (21)	FL (27)	OH (20)	FL
MI (17)	PA (21)	OH (20)	FL (27)	OH
OH (20)	FL (27)	MI (17)	PA (21)	FL
OH (20)	FL (27)	PA (21)	MI (17)	FL
OH (20)	MI (17)	FL (27)	PA (21)	FL
OH (20)	MI (17)	PA (21)	FL (27)	PA
OH (20)	PA (21)	FL (27)	MI (17)	FL
OH (20)	PA (21)	MI (17)	FL (27)	MI
PA (21)	FL (27)	MI (17)	OH (20)	Fl
PA (21)	FL (27)	OH (20)	MI (17)	FL
PA (21)	MI (17)	FL (27)	OH (20)	FL
PA (21)	MI (17)	OH (20)	FL (27)	OH
PA (21)	OH (20)	FL (27)	MI (17)	FL
PA (21)	OH (20)	MI (17)	FL (27)	MI

- **Solution (cont.):**

3. Florida has the swing in 12 cases, Michigan has the swing in 4 cases, Ohio has the swing in 4 cases, and Pennsylvania has the swing in 4 cases.

   So the index for Florida is $12/24 = 1/2$ or 50%,

   Michigan: $4/24 = 1/6$ or about 16.67%,

   Ohio: $4/24 = 1/6$ or about 16.67%,

   Pennsylvania: $4/24 = 1/6$ or 16.67%.

The Shapley-Shubik Index of a given voter is calculated as the fraction of all permutations of the voters in which that voter is the swing.

Learning Objectives:

- Show there is no perfect voting system if there are three or more candidates
- Touch on some of the different voting systems used
- Understand concepts such as:
  - Instant Runoff Voting
  - The Spoiler Effect
  - Plurality Casting
  - Condorcet Winner
  - Independence of Irrelevant Alternatives

8.2 Voting systems: How do we choose a winner?

- **Voting System:** A set of rules under which a winner in an election is determined.
- **Plurality Voting:** The system of voting in which the candidate that receives more votes than any other candidate is the winner.

- **Example:** Of four candidates and 100 votes, what is the smallest number of votes needed to win?
- **Solution:** If the four candidates have an equal number of votes, they would have $100/4 = 25$ each. So a candidate could have a plurality with as few as 26 votes.

8.2 Voting systems: How do we choose a winner?

- **Spoiler:** A candidate who has no realistic chance of winning but whose presence in the election affects the outcome.
- **Example:** In the 2000 election, the Florida vote tally for the candidates was as follows:

Candidate	Votes
George W. Bush	2,912,790
Al Gore	2,912,253
Ralph Nader	97,488
Others	40,579

"Others" includes five other candidates and write-in votes.

- **Example (cont.):**

1. What percent voted for Bush? Did anyone achieve majority?
2. Suppose the election only included the top three candidates holding their votes but the votes for "others" were additionally distributed to these three. Could anyone achieve majority?
3. Assume Nader did not participate, "Others" remain, and all voters for Nader are redistributed or don't vote. How many more of Nader's votes must Gore obtain than Bush to win?
4. Exit polls indicate that 21% of Nader voters would have voted for Bush and 47% of them would have voted for Gore. It is estimated that $14 - 17\%$ of Nader votes would have gone to Bush and $32 - 40\%$ would have gone to Gore. Does this indicate that Nader was a spoiler?

- **Solution:**

1. Bush received $2{,}912{,}790$ of the $5{,}963{,}110$ total votes, which is $2{,}912{,}790/5{,}963{,}110$ or 48.8%, which is not a majority.
2. $5{,}963{,}790/2 = 2{,}981{,}555$, so a majority is $2{,}981{,}556$ votes. That is $68,766$ votes more than were cast for Bush.
3. Bush received 537 more votes than Gore. So Gore would need 538 more Nader votes than Bush.
4. 21% of the Nader votes is $20{,}472$ and 32% is $31{,}196$.

   This gives Bush a total of 2,933,262 and Gore a total of $2{,}943{,}449$ votes.

   If Nader had not been on the ballot, the outcome of the election would have been different.

   Therefore Nader could have reasonably been considered to be a spoiler for the election.

8.2 Voting systems: How do we choose a winner?

- **Preferential Voting System:** Systems in which voters express their ranked preferences between various candidates, usually with a **ranked ballot** which is used to avoid several rounds of voting and the voter lists his or her candidate preferences. Two examples follow.
- **Top-Two Runoff System:** If no candidate receives majority, there is a new election with only the two highest vote-getters.
- **Elimination Runoff System:** If no candidate receives majority, the lowest vote-getter is eliminated and a vote is taken again among those who are left. This repeats until a majority is reached.

8.2 Voting systems: How do we choose a winner?

- **Example:** Consider the following ranked ballot outcome for 10 voters choosing among three candidates:

Rank	**4 voters**	**4 voters**	**2 voters**
First Choice	Alfred	Gabby	Betty
Second Choice	Betty	Alfred	Gabby
Third Choice	Gabby	Betty	Alfred

- Determine the winner under the elimination runoff system.

- **Solution:**
  - No candidate has first-choice majority, Betty has the least so she is eliminated from the first round.

Rank	**4 voters**	**4 voters**	**2 voters**
First Choice	Alfred	Gabby	~~Betty~~
Second Choice	~~Betty~~	Alfred	Gabby
Third Choice	Gabby	~~Betty~~	Alfred

  - With Betty eliminated, the table is now as follows:

Rank	**4 voters**	**4 voters**	**2 voters**
First Choice	Alfred	Gabby	Gabby
Second Choice	Gabby	Alfred	Alfred

  - Now the first-choice votes are 4 for Alfred and $4 + 2 = 6$ for Gabby. In this runoff, Gabby has majority and is the winner.

8.2 Voting systems: How do we choose a winner?

- **Borda count:** A method of ranked balloting that assigns for each ballot: 0 points to the choice ranked last, 1 point to next higher choice, and so on. The **Borda winner** is the candidate with the highest Borda count.
- **Example:** To decide on food, five friends mark ranked ballots by preference, using the table:

	Pizza	Tacos	Burgers
Ballot 1	2	1	0
Ballot 2	2	1	0
Ballot 3	2	1	0
Ballot 4	0	2	1
Ballot 5	0	2	1

1. Did one of the foods receive a majority for first-choice?
2. Use the Borda count to determine which food should be ordered.

- **Solution:**

1. First-place votes are indicated by the number 2. There were three first-place votes for pizza, which is a majority of the five first-place votes.
2. The Borda count is:

For pizza: $2 + 2 + 2 + 0 + 0 = 6$

For tacos: $1 + 1 + 1 + 2 + 2 = 7$

For burgers: $1 + 1 = 2$

According to the Borda count, the group should order tacos. This is true in spite of pizza receiving majority.

This lends to our understanding that no voting system is perfect with three or more candidates.

8.2 Voting systems: How do we choose a winner?

- **Example:** The five finalists for the 2009 Heisman Trophy follow:

Player	1st-Place Votes	2nd-Place Votes	3rd-Place Votes
T. Gerhart (Stanford)	222	225	160
M. Ingram (Alabama)	227	236	151
C. McCoy (Texas)	203	188	160
N. Suh (Nebraska)	161	105	122
T. Tebow (Florida)	43	70	121

- Determine the Borda counts and the Borda winner.

- **Solution:**

Player	1st-Place Votes	2nd-Place Votes	3rd-Place Votes
T. Gerhart (Stanford)	222	225	160
M. Ingram (Alabama)	227	236	151
C. McCoy (Texas)	203	188	160
N. Suh (Nebraska)	161	105	122
T. Tebow (Florida)	43	70	121
Borda count value	**2**	**1**	**0**

Borda count for T. Gerhart $= 222 \times 2 + 225 \times 1 + 160 \times 0 = 669$

M. Ingram $= 227 \times 2 + 236 \times 1 + 151 \times 0 = \mathbf{690}$

C. McCoy $= 203 \times 2 + 188 \times 1 + 160 \times 0 = 594$

N. Suh $= 161 \times 2 + 105 \times 1 + 122 \times 0 = 427$

T. Tebow $= 43 \times 2 + 70 \times 1 + 121 \times 0 = 156$

8.2 Voting systems: How do we choose a winner?

- **Example:** Consider the following ranked ballot outcome for 100 voters choosing from options A, B, C, D:

Rank	28 votes	25 votes	24 votes	23 votes
1st Choice	A	B	C	D
2nd Choice	D	C	D	C
3rd Choice	B	D	B	B
4th Choice	C	A	A	A

1. Who wins under plurality voting?
2. Who wins under the top-two runoff system?
3. Who wins under the elimination runoff system?
4. Who wins under the Borda count system?

- **Solution:**

1. Under plurality voting only first-choice picks are considered. In that case, candidate A has the most first- choice votes with 28/100.
2. For a runoff with only the top two candidates, A and B, C and D are eliminated from the table, as below:

Rank	28 Votes	25 Votes	24 Votes	23 Votes
Adjusted 1st Choice	A	B	B	B
Adjusted 2nd Choice	B	A	A	A

- In this runoff, B has 72 first-choice votes; this is clearly a majority, so B is the winner.

- **Solution (cont.):**

3. In the first round of elimination runoff, D has the fewest votes and is eliminated, the table now follows:

Rank	28 votes	25 votes	24 votes	23 votes
1st Choice	A	B	C	C
2nd Choice	B	C	B	B
3rd Choice	C	A	A	A

- Now B has the fewest votes and is eliminated:

Rank	28 votes	25 votes	24 votes	23 votes
1st Choice	A	C	C	C
2nd Choice	C	A	A	A

- C wins in this runoff by a majority of 72 votes.

- **Solution (cont.):**

4. The Borda count incorporated into the original table:

Rank	**Borda Value**	28 votes	25 votes	24 votes	23 votes
1st Choice	**3**	A	B	C	D
2nd Choice	**2**	D	C	D	C
3rd Choice	**1**	B	D	B	B
4th Choice	**0**	C	A	A	A

So the Borda count for each candidate is:

$$A = 28 \times 3 + 25 \times 0 + 24 \times 0 + 23 \times 0 = 84$$

$$B = 28 \times 1 + 25 \times 3 + 24 \times 1 + 23 \times 1 = 150$$

$$C = 28 \times 0 + 25 \times 2 + 24 \times 3 + 23 \times 2 = 168$$

$$D = 28 \times 2 + 25 \times 1 + 24 \times 2 + 23 \times 3 = 198$$

Hence D is the winner based on Borda count.

Common Preferential Systems

- **Plurality:** The candidate with the most votes wins.
- **Top-two runoff:** If no one garners a majority of the votes, a second election is held with the top two getters as the only candidates.
- **Elimination runoff:** Successive elections are held where the candidate with the smallest number of votes is eliminated. This continues until there is a majority winner.
- **Borda count:** Voters rank the candidates first to last. The last-place candidate gets 0 points, the next 1 point, and so on. The candidate with the most points wins.

- A **Condorcet Winner** is a candidate who beats each of the other candidates in a 1– 1 election.

- **Example:** Suppose in an election there are 7 voters and three candidates, A, B, and C; the voters' preferences follow:

Preferences	**3 voters**	**2 voters**	**2 voters**
1st Choice	A	C	C
2nd Choice	B	B	A
3rd Choice	C	A	B

Is there a Condorcet winner? If so, which candidate?

- **Solution:**

  Find the results of each head-to-head contest:

  - Consider A and B: there are $3 + 2 = 5$ voters who rank A over B, and only 2 who rank B over A. So A wins versus B.

  - The results of the other head-to-head contests are:
    - A and C: C wins by 1 (4 to 3)
    - B and C: C wins by 1 (4 to 3)

  - There is a Condorcet winner—it is candidate C.

8.2 Voting systems: How do we choose a winner?

- **Example:** In an election there are seven voters and candidates A, B, C, and D. The tally of ranked ballots follows:

Voter:	**1**	**2**	**3**	**4**	**5**	**6**	**7**
1st Choice	A	A	B	C	D	A	C
2nd Choice	B	D	A	B	B	D	B
3rd Choice	C	B	C	A	A	B	A
4th Choice	D	C	D	D	C	C	D

1. Who wins the plurality system?
2. Who wins the top-two runoff system?
3. Who wins in the elimination runoff system?
4. Who wins the Borda count?
5. Is there a Condorcet winner? If so, which candidate?

- **Solution:**

1. In a plurality voting, only first choices are considered. Candidate A has 3 votes, B has 1, and C has 2 votes. A has the most, so A wins.
2. The first-place winner is A. Candidate C is second with 2 votes. In a runoff with A and C, A wins with 5 votes to 2.
3. Because B and D only get one vote each, they are eliminated and A wins.
4. The Borda count is as follows: $\mathrm{A} = 3 \times 3 + 1 \times 2 + 3 \times 1 = 14$

$$\mathrm{B} = 1 \times 3 + 4 \times 2 + 2 \times 1 = 13$$
$$\mathrm{C} = 2 \times 3 + 2 \times 1 = 8$$
$$\mathrm{D} = 1 \times 3 + 2 \times 2 = 7$$

So, A wins the Borda count.

5. There is a Condorcet winner. The head-to-head outcomes are as follows:

B beats A (4 to 3) B beats C (5 to 2) B beats D (4 to 3)

So, B is the Condorcet winner.

- The **Condorcet winner criterion** says that if there's a Condorcet winner, then he or she should be the winner of the whole election.
- The condition of **Independence of irrelevant alternatives** states: Supposing candidate A wins an election and B loses, and another election follows in which no voter changes their preference concerning A and B, B should still lose to A no matter what happens concerning the other candidates.
- **Arrow's Impossibility Theorem:** If there are three or more candidates, there is no voting system (other than dictatorship) for which the Condorcet winner criterion and the Independence of irrelevant alternatives hold.